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Decentralized Control Problems in Neumann-Economies*

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1. LITERARY ANTECEDENTS

Our paper has been born from the cross breeding of three trends in research.

1. *Mathematical modeling of the control mechanism of economic systems.* The first experiments in this direction were made by the neoclassical school in the 19th century by Cournot [7] and Walras [34], then, during the 20th century renaissance of the general equilibrium theory, by Samuelson [29], Arrow and Debreu [9] and their followers. Recently, several authors have attempted to part, partially or entirely, with the neoclassical schemes of reasoning in the mathematical modeling of the control of economic systems. Important pioneers were Hurwicz [13], as well as Marschak and Radner [24].

2. *The dynamic Leontief and the Neumann models.* (See, e.g., [4, 20, 27]). As opposed to the research trend reviewed in paragraph one, these were intended originally to be only models of the *real* sphere and were not linked to the description of *control.* The first steps towards analyzing the control of the Leontief and the Neumann economics were made by Sargan [30], Leontief [21], Lovell [22], and Jorgenson [14].

3. *Application to economics of mathematical control theory and dynamic programming.* Up to now these have been mainly used for the formalization of Keynesian macroeconomics, capital theory, and the theory of growth. (See, e.g., Tustin [31], Lange [19], Dorfman [10],

* J. Kornai wrote a longer study [18], in which he outlined a few main ideas of the present paper in 1973, during his visit with the Institute of Mathematical Studies in the Social Sciences, Stanford University, California, and with the Institute of Mathematical Economics, Bielefeld University, GFR. Thanks are due to the two institutions for promoting the work. Later on the authors jointly continued the research in Budapest. We are indebted to the Institute of Economics, Hungarian Academy of Sciences for support, and to B. Martos, I. Dancs, Z. Kapitány and an anonymous referee of this Journal for valuable comments.

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Radner [28], and Arrow and Kurz [2].) Also, the control of econometric simultaneous equation systems with the aid of control theory devices is known (see Chow [6]). It has also been used to study some dynamic planning problems, e.g., in the study by Kendrick [15]. But up to now it has been used only exceptionally (e.g., in McFadden [23]) for modeling the control mechanism of economic systems.

The amalgamation of the three trends was attempted in a study written by one of the authors of the present article, J. Kornai, with coauthor B. Martos [16], [17] about autonomous (vegetative) control, and in several other papers improving upon the Kornai–Martos model: Dancs, Hunyadi, and Sivák [8], Virág [33], and Bródy [5]. The present article also belongs to this series. Its aim, like that of the earlier studies, is to "cross breed" the three trends: to foster the researches of the first trend, that is of the abstract modeling of the control mechanisms while using the apparatus and results of the second trend (dynamic Leontief and Neumann models) and of the third (control theory). This crossbreeding involves, of course, not only advantages, but also strong restrictions. We are compelled to use in combination the strong assumptions applied on the one hand in the literature of the Leontief and the Neumann models, and by mathematical control theory, on the other hand.

Having briefly reviewed the literary antecendents of our paper, we are going to review the models and the conclusions that can be drawn with their aid. In some places we will make references to literary sources also under way, and, in a later part of the study a separate section will be devoted to comparing our results with those of other papers.

2. The Model[1]

General Remarks

Exclusively *dynamic systems* will be investigated. Time is an integer variable: $t = 0, 1, 2, \ldots$. The time unit is called a *period*. A stock variable always refers to a state at the beginning of the period (e.g., initial stock), while the flow takes place in the course of the period.

[1] The main principles of notation: *Matrices* will be denoted by italic capital letters; their elements by the same lowercase letters and double subscript. *Vectors* are denoted by italic lowercase letters, their components by the same letter with a subscript. Transposition is denoted by a prime. A *diagonal matrix* is denoted by the vector in the main diagonal between the symbols $\langle\ \rangle$. The logical product of two matrices (by elements) is denoted by $\otimes$ and their inequality by $\geqq$ and $\geqslant$, where the former does not exclude that every pair of elements should be equal, while the latter excludes it. 1 stands for unity as well as for the summation vector of dimension n.

The *real sphere* of the economic system comprises the variables of the basic real processes (production, productive consumption, personal and public consumption, turnover) and the relationships existing between them. On the theoretical plane the *control sphere* controlling the real sphere is distinguished from it. In some notations reference is made to the real sphere with an R and to the control sphere by using C.

The economy has n kinds of *products*; they are produced by n *sectors* of the economy.

Two kinds of economies will be examined. The first is called a *Neumann-economy* because, as regards its essential features, it corresponds to an economy having a constant structure, increasing at a steady rate and proceeding around a Neumann-path. The other one will be called an ϵ-asymptotic Neumann-economy, or, for short, an *asymptotic economy*.[2] By simplifying the explanation somewhat we can say the latter is such a dynamic Leontief system, coming very near to a Neumann-economy with a constant structure. A more precise definition of the two economies will emerge from the detailed discussion of the assumptions and theorems. For the most part, the assumptions used in modeling the two economies will be identical. In denoting the alternative assumptions and theorems which differ in the two economies, the distinctions N (Neumann-economy) and A (asymptotic economy) will be applied.

Variables

The variables used in the model are the following.

$r(t)$ = the vector of *production* with n components. The variable r_j is the output of sector j.

$Y(t)$ = the matrix of *purchases*, of the size $n \times n$. The variable y_{ij} is the quantity of the product i purchased by the sector j.

$w(t)$ = the vector of the *output stock* with n components. The variable w_j is the stock accumulated by sector j from its own product in its own inventory.

$V(t)$ = the matrix of the *input stock*, of the size $n \times n$. The variable v_{ij} is the stock of product i accumulated in the warehouse of sector j.

$S(t)$ = the *slack input stock*[3] of the size $n \times n$. The variable s_{ij} is the part of the total input stock v_{ij} above the technologically necessary quantity. This definition will still be reverted to.

[2] Of course, the first type is a special case of the second one.

[3] The denomination "slack" indicates the part of the resources which is not utilized. The name "slack" is itself a neutral word, it does not involve any value judgement. It contains the *reserves* deliberately built up for balancing random disturbances, as well as the *surpluses*, the residuals unjustified from the point of view of rational management.

The Real Structure

There are two kinds of technological coefficient matrices in the model:

$A(t)$ = the *matrix of current input coefficients*, of the size $n \times n$. It corresponds to the matrix A well known from the static and dynamic Leontief models. Scrapping is included in the current input coefficients. Scrapping ratios are given exogenously.

$B(t)$ = the *matrix* of the technological input stock coefficients, i.e., of the coefficients denoting the technological engagement of assets, of the size $n \times n$. As a matter of fact, this corresponds to the matrix B of the dynamic Leontief model, but with an important difference. It comprises only the engagement of assets *technologically indispensable for production*. If, therefore, sector 8 is the producer of power generators and sector 2 is the producer of electric energy, $b_{8,2}$ denotes the quantity of generators directly participating in producing a unit of electric energy, *without the reserve* generators. Accordingly:

$$v_{ij}(t) = b_{ij}(t)\, r_j(t) + s_{ij}(t),$$

that is, the total input stock is composed of two parts, the *technological input stock* $b_{ij}(t)\, r_j(t)$ and the *slack input stock* $s_{ij}(t)$, the later serving as reserve.

Notation. We denote by J_t the following set: $J_t = \{(i, j); a_{ij}(t) = 0\}$. Of course, $b_{ij}(t) = 0$ if and only if $(i, j) \in J_t$. As usual, $\bar{J}_t$ stands for the complementum of J_t: $\bar{J}_t = \{(i, j); a_{ij}(t) > 0\}$.

We assume that J_t is time-invariant: $J_t = J$ for all t's. In addition, we assume that $v_{ij}(t) = 0$, $s_{ij}(t) = 0$, and $y_{ij}(t) = 0$ if $(i, j) \in J$. In this paper, for example, $V(t)$ is referred to as $\bar{J}$-positive, if $v_{ij}(t) > 0$; $(i, j) \in \bar{J}$, etc.

The pair of the two coefficient matrices, $\{A(t), B(t)\}$, is called the *real structure* of the system.

The Product Balance

The functioning of the real sphere of the model is easier to understand if, using scalar notation, the product balance of sector i in period t is surveyed:

$$r_i(t) = \sum_{j=1}^{n} a_{ij}(t)\, r_j(t) + w_i(t + 1) - w_i(t) + \sum_{j=1}^{n} [v_{ij}(t + 1) - v_{ij}(t)]. \quad (1.1)$$

Production is used for three purposes (for the sake of a simpler interpretation, let us assume that both production and stock are growing):

first term: current inputs into production;

second and third term: accumulation necessary for increasing the output stocks;

fourth term: the accumulation necessary to increase the input stocks.

The total of the output stocks and slack input stocks of the economy is called *buffer stock* or *slack*. Under ideal conditions (functioning completely without friction and disturbance) the economy could operate exclusively with the aid of the technological input stocks, without buffer stocks (slacks).

Assumptions About the Real Sphere

Let us sum up the assumptions about the real sphere.

R.1. *Closed Leontief-economy.* A product is turned out by a single sector and conversely, every sector releases but a single product. There is no substitution between inputs and no choice between alternative technologies. There are no joint products. There are no external resources nor final consumption. The consumption of the "labor sectors" of the economy appears as input, their work as output, and, therefore, they are not distinguished in the model from the other "common" sectors. The economy is irreducible.

R.2. *Homogenous linear input functions.* The current input into production and the technologically necessary engagement of assets are homogeneous linear functions of the volume of output.

R.3. *Exogenous scrapping ratios.* The scrapping ratio of the assets technologically engaged in production per period is exogenously given.

R.4. *Productivity of the system.* The system is capable of growth. Beyond current inputs there remains a surplus for the building up and expanding of stocks (output stocks, technological and slack input stocks). In other words, the spectral radius of the matrix $A(t)$ is uniformly smaller than unity for all t's: $\sigma\{A(t)\} < 1 - \kappa$ where κ is a positive number less than unity, which is independent of t.

R.5.N. *The real structure is time-invariant.* The coefficient matrices of the real structure are constant over time: $A(t) = A$, $B(t) = B$.

R.5.A. *The real structure is asymptotic.* The coefficient matrices of the real structure closely approach, after the lapse of a finite time, some *basic structure*.

More precisely, for each time-invariant real structure $\{A, B\}$ there exist an appropriately small number $\epsilon > 0$ and an appropriately large integer T, with the neighborhood of time-variant real structures $\{A(t), B(t)\}$

defined as follows: If $t > T$, then $\| A(t) - A \|$, $\| B(t) - B \| < \epsilon$, where $\| \cdot \|$ denotes some norm of a matrix (e.g., the sum of the absolute values of the elements).

A time-variant real structure is called asymptotic if it is contained in the neighborhood of at least one time-invariant real structure.

Our figure illustrate this ϵ-asymptotic property with the example of the coefficient $a_{ij}(t)$.

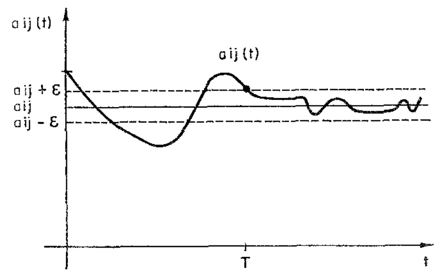

FIG. 1. Development of the input–output coefficients over time in an asymptotic economy.

Normative Control

Proceeding from the review of the real sphere to describing the control sphere, first of all the basic idea of *normative control* will be expounded.

It is assumed that for the decision making units of the economy (in our model the sectors) some *norms* are given which serve as guidelines in their behavior. In a dynamic system the norms appear perhaps in the form of *normative paths*. These are paths indicating the normative direction of the movement of the system.

The norms are developing as a result of historical and social processes and are fixed by conventions. In our paper the *origin* of the norms is not dealt with; this will be done in other studies. Here we simply assume they are a "given."

In this paper in the case of a Neumann economy the norms can be *computed* in a definite manner. But this is only a "trick" of the analysis of the system. It is not linked to any such institutional assumption as if in reality anyone "worked out" the norms of the system. In real life the norms are shaped tentatively by anonymous social processes.

Now if the norms (or the normative paths) are given, the decision makers may observe the differences between the actual and the normative values of the variables. Knowing this and counteracting the deviation they interfere and channel the system back towards the norms. Normative regulation takes place according to the principle of *negative feedback*, according to the following general *control formula*:

$$u(t) - u^*(t) = \varphi[t, x(t) - x^*(t)], \qquad \varphi(0) = 0, \tag{1.2}$$

$\varphi(t, \hat{x})$ monotonically decreases in $\hat{x}$; where $u(t)$ is the *control variable* of the system and $x(t)$ is the *state variable*. The asterisk distinguishes the normative value of the variable from its actual value without the asterisk. The form of the dependance φ must, of course, be specified in each concrete model.

In our model the production $r(t)$ and the purchase $Y(t)$ are the control variables, while the output stock $w(t)$ and the input stock $V(t)$ are the state variables.

Also a shortened notation will be introduced:

$$\begin{aligned} \hat{u}(t) &= u(t) - u^*(t), \\ \hat{x}(t) &= x(t) - x^*(t). \end{aligned} \tag{1.3}$$

These variables are called *deviation variables*.

Thus also the general control formula can be written in the form of

$$\hat{u}(t) = \varphi[t, \hat{x}(t)]. \tag{1.4}$$

Assumptions About the Control Sphere

Let us sum up the assumptions relating to the control sphere.

C.1. *Normative control.* The normative paths are feasible and given for the system: $r^*(t)$, $Y^*(t)$, $w^*(t)$, and $V^*(t)$.

Moreover, these variables as well as $S^*(t)$ are $\bar{J}$-bounded away from zero for all t's, and there exists a positive number κ which is smaller than every $\bar{J}$-coordinate of these variables for all t's.

C.2.N. *The normative path is a Neumann-path.*[4] A Neumann-path is denoted as follows:

$$\begin{aligned} \bar{r}(t) &= r_0\lambda_0{}^t, \\ \bar{Y}(t) &= Y_0\lambda_0{}^t, \\ \bar{w}(t) &= w_0\lambda_0{}^t, \\ \bar{V}(t) &= V_0\lambda_0{}^t, \end{aligned} \tag{1.5}$$

[4] We do not consider its parallel, C.2.A, where the normative path is an asymptotic Neumann-path. See Theorem 1.

where λ_0 is the Neumann growth coefficient: $\lambda_0 > 1$. According to the assumption, the Neumann-path serves as a normative path, that is:

$$\begin{aligned} r^*(t) &= \bar{r}(t), \\ Y^*(t) &= \bar{Y}(t), \\ w^*(t) &= \bar{w}(t), \\ V^*(t) &= \bar{V}(t). \end{aligned} \tag{1.6}$$

Let us denote the matrix of the norms of the slack input stock per output by F and that of the norms of the output stock per production by $\langle g \rangle$:

$$\bar{S}(t) = F\langle \bar{r}(t) \rangle \quad \text{and} \quad \bar{w}(t) = \langle g \rangle \, \bar{r}(t), \tag{1.7}$$

and let us denote by H the matrix of the norms of buffer stocks: $H = F + \langle g \rangle$.

C.3. *Decentralization of the decisions.* With given norms decision is fully decentralized. Sector j decides on production r_j and for every i on the purchase y_{ij}.

There is a further degree of decentralization in the economy. In each sector production decision r_j and every purchase y_{ij} are made independently of each other. We may visualize the situation as if there were a production department responsible for production decisions and n purchase departments responsible for purchase decisions and these $n + 1$ departments were operating independently of each other.

C.4. *No memory.* Our decentralized control does not avail itself of a memory; it does not use as information the values of the variables taken in earlier periods.

C.5. *Decentralization of information.* With given norms information is completely decentralized. Sector j observes for making decisions only its own state variables. Let us call *autonomous* (*vegetative*) *control* the control scheme in which the assumptions C.3 and C.5 are asserted in combination.

Similarly to the decentralization of decision (see C.3) information is also decentralized within the sector among the production department and the purchase departments.

C.6. *Exclusivity of stock signal.* Assumption C.5 is realized in such a manner that the production department in sector j observes exclusively its own output stock w_j, and purchase department in sector j responsible for purchasing commodity i observes exclusively its own input stock v_{ij}.

C.7. *Linearity of the control rule.* Control is secured with the aid of a linear formula. Its general form is $\hat{u}(t) = \alpha\hat{x}(t)$, where $\alpha = \langle\alpha_i\rangle$ is the diagonal matrix of *the speeds of adjustment.*[5] In formulas (2.3) and (2.4) below, $\langle d\rangle$ and E figure as matrices of speeds of adjustment.

C.8. *Homogeneity of adjustment speeds.* $\alpha_1 = \alpha_2 = \cdots = \alpha_m$.

Summary of the Model

Our dynamic system is described with the ensemble of the following four equations:

Output stock

$$w(t+1) = w(t) - Y(t)1 + r(t). \tag{2.1}$$

Slack input stock

$$V(t+1) = V(t) - A(t)\langle r(t)\rangle + Y(t). \tag{2.2}$$

Control of production

$$r(t) = r^*(t) - \langle d\rangle[w(t) - w^*(t)]. \tag{2.3}$$

Control of purchases

$$Y(t) = Y^*(t) - E \otimes [V(t) - V^*(t)]. \tag{2.4}$$

We have altogether $2n + 2n^2$ equations.

Equations (2.1)–(2.2) give the *laws of motion* and (2.3)–(2.4) give the *controls* of the system. The former deduce the new state variables of the period $(t + 1)$ with the aid of the control and state variables of the period t. The latter deduce from the deviations between the actual and normative values of the state variables the deviations of the control variables from their normative values.

3. Theorems

Below we are going to review theorems, together with some short economic comments. The mathematical proofs of the theorems are to be found in the Appendix.

THEOREM 1[6] (The existence of a normative path). *Under Assumptions* R.1–R.4 *there exists a zone of feasible paths satisfying Assumption* C.1.

[5] In Control Theory α is often called "control gain."

[6] We prove Theorem 1 without R.5.A, in its full generality.

THEOREM 1.N (The existence and the uniqueness of the normative Neumann-path). *Under Assumptions* R.1–R.4 *and* R.5.N *there exists a unique Neumann-path which satisfies Assumption* C.1 *and* C.2.N *and it is provided as a dominant solution of an eigenvalue–eigenvector problem.*

Remark. A word of explanation will be in order to clarify why C.1 is regarded as an *assumption* and Theorem 1 as a *conclusion.* In C.1 it is emphasized, that a feasible normative path is *given*, determined by historical and social processes, while in Theorem 1 it is emphasized that there exists a normative path which is *feasible* under the assumptions describing the technological conditions of the real sphere.

The relation between Assumptions C.1, C.2.N and Theorem 1.N is similar.

THEOREM 2.N (Buffer stock and growth rate). *Under Assumptions* R.1–R.4, R.5.N *and* C.2.N *let us compare economies* 1 *and* 2, *which are fully identical except for the norms of the buffer stock*: $h_{ij}^{(1)} \geqq h_{ij}^{(2)}$ *for every* (i, j), *and at least one component is definitely larger in economy* 1 *than in economy* 2. *In this case the growth coefficient of economy* 1 *is smaller than that of* 2: $\lambda_0^{(1)} < \lambda_0^{(2)}$.

Remark. With this theorem we state that in the world of our deterministic model the increase of the buffer stock norms slows down the growth of the economy. In real life the increasing of the buffer stock norms also has some advantages: it may help to fight the unexpected disturbances of production, make adaptation smoother, etc. But we cannot prove these advantages in this paper.

DEFINITION. The economy described in (2.1)–(2.4) is *stable*[7] if starting from an arbitrary initial point $[w(0), V(0)]$, all deviation variables, $\hat{w}(t)$, $\hat{V}(t)$, $\hat{r}(t)$, and $\hat{Y}(t)$, converge to zero as t converges to infinity.

The economy is *relatively stable* if starting from an arbitrary initial point $[w(0), V(0)]$, all relative deviation variables, $\hat{w}_j(t)/w_j^*(t)$, $\hat{v}_{ij}(t)/v_{ij}^*(t)$, $\hat{r}_j(t)/r_j^*(t)$, and $\hat{y}_{ij}(t)/y_{ij}^*(t)$, converge to zero as t converges to infinity.

Remarks. Since all normative variables are bounded away from zero, stability implies relative stability.

For the time being, up to the discussion of Theorem 5.A, we will neglect the question of positivity of variables $w(t)$, $V(t)$, $S(t)$, $r(t)$, and $Y(t)$. Since our economy is linear, local stability is equivalent to global stability.

[7] In the theory of difference equations this property is often called *asymptotic stability*.

THEOREM 3.A[8] (Stability). *Conditions of stability of the system* (2.1)–(2.4) *under Assumptions* R.1–R.4, R.5.A, C.1, *and* C.3–C.7.

(i) Sufficient condition. *The system is stable if the decentralized feedback is* damped:

$$0 < d_j,\ e_{ij} \leqq 1 \qquad 1 \leqq i, j \leqq n.$$

(ii) Necessary and sufficient condition. *Under the additional Assumption* C.8, *where* $\delta = d_j = e_{ij}$ $(1 \leqq i, j \leqq n)$, *the system is stable if and only if*

$$0 < \delta < 2/[1 + \sigma^{1/2}(A)].$$

THEOREM 3.N (Relative stability: Necessary and sufficient condition). *Under Assumptions* R.1–R.4, R.5.N, C.1, C.2.N, *and* C.3–C.8, *the system* (2.1)–(2.4) *is relatively stable if and only if*

$$-(\lambda_0 - 1)/(1 + \sigma^{1/2}(A)) < \delta < (\lambda_0 + 1)/(1 + \sigma^{1/2}(A)).$$

Remark. It is trivial that our system is stabilizable in a *centralized* way, under the above mentioned assumptions, without R.5.A, by the following rule:

$$\hat{r}(t) = -[I - A(t)]^{-1}[\hat{w}(t) + \hat{V}(t)1],$$
$$\hat{Y}(t) = -A(t)[I - A(t)]^{-1}[\hat{w}(t) + \hat{V}(t)1] - \hat{V}(t).$$

In fact, all deviation variables become zero if $t \geqq 1$.

The major innovation in our analysis is the statement that the stabilization of a strongly coupled system like (2.1)–(2.4) can be assured in a *decentralized* way as well. We used Assumption R.5.A for that purpose. It is still unanswered if our system is stabilizable in a decentralized way, without R.5.A.

DEFINITION. We call systems with identical real structures $\{A, B\}$ *similar* economies.

THEOREM 4 (Adjustment speeds and convergence rates.[9]) *We compare similar economies, differing only in adjustment speeds, under Assumptions* R.1–R.4, R.5.N, C.1 *and* C.3–C.7. *We consider the following cases*:

[8] Of course, Theorem 3.A is valid for a Neumann economy as well, since a Neumann-economy is *a fortiori* an asymptotic economy. The same remark applies to Theorem 5.A.

[9] The term "convergence rate" is well known in the literature of difference equations. For special definition in our case see the proof of Theorem 3.A.

(i) *We compare two similar economies*: 1 *and* 2. *The adjustment speeds are damped*: $0 < d_j^{(1)} \leqq d_j^{(2)} \leqq 1$ *and* $0 < e_{ij}^{(1)} \leqq e_{ij}^{(2)} \leqq 1$ *for every j and* $(i, j) \in \bar{J}$, *and at least one adjustment speed is smaller in economy* 1 *than in economy* 2. *In this case the convergence rate of the economy* 1 *is smaller than that of* 2: $1/\hat{\lambda}^{(1)} < 1/\hat{\lambda}^{(2)}$.

(ii) *We consider all similar economies with damped adjustment speeds. The convergence rate is maximal if and only if*

$$d_j = 1 = e_{ij} \quad \text{for} \quad j = 1, 2, \ldots, n \quad \text{and} \quad (i, j) \in \bar{J}.$$

(iii) *We consider all similar economies with homogenous adjustment speeds* (C.8). (*In that case we do not assume, as in* (ii), *that the adjustment speeds are damped.*) *The convergence rate is maximal if and only if*

$$\delta = 1.$$

Remarks. Alternative assumptions (ii) (*damped* adjustment speeds) and (iii) (*homogenous* adjustment speeds) lead to the same result: we achieve fastest convergence by adjustment speeds all equal to unity. The maximal convergence rate is equal to $1/\sigma^{1/2}(A)$.

Our theorem is independent of the choice of the normative path, and, *a fortiori*, of F and $\langle g \rangle$.

We apply neither suffix N, nor suffix A denoting Theorem 4, for the following reasons: We assume a time-invariant real structure, i.e., the system is a Neumann-economy in this respect. We do not apply, however, Assumption C.2.N, i.e., we do not assume that the normative path is a Neumann-path. In this respect the Theorem is more general.

DEFINITION. The system is *viable* if none of its variables is negative, and in each period at least one production variable is positive. We call the *viable initial neighborhood* the set of all initial points which provide viable systems.

THEOREM 5.A (Viability). *Under the Assumptions* R.1–R.4, R.5.A, C.1, *and* C.3–C.7, *in every* (*relatively*) *stable economy we can find a viable initial neighborhood.*

Remarks. The theorem holds for both the Neumann-economy and the asymptotic economy. For the Neumann-economy, relying on the additional Assumption C.2.N, we have a simple constructive method for determining a viable initial neighborhood. (See the proof of the Theorem.)

The common practical interpretation of Theorems 3.A and 5.A is the following: Under definite conditions the control described in this paper is viable and is suited for securing (relative) stability.

4. About the Economic Interpretation of the Theorems

Elementary Quantitative Adaptation

Like its predecessors, the Kornai–Martos model and the models of Dancs–Hunyadi–Sivák and of Virág, the model described in this paper does not claim to be a *general* model of the regulation of the economy.

The regulation of economic systems is a complex problem, a task performed in combination by various control mechanisms and submechanisms. A part of these control mechanisms control primitive, short-term quantitative adaptation processes, with very simply decentralized decision and information structures. The Kornai–Martos article called this type of control "autonomous" or "vegetative" control. Our paper intends to contribute to the analysis of exclusively such "elementary," "primitive," "autonomous," "vegetative" control mechanisms.

At the same time, in every economy there are also mechanisms operating, which fulfil more complicated control tasks. Such are, for example, the qualitative development of production and consumption, the introduction of new products, the combination of various resources, and, in this context, the choice between alternative technologies; decision on the rate of accumulation, long-term adaptation to lasting changes, and so on. Obviously, the mechanism described in the present study is not suited for fulfilling the more complicated control functions.

The World of Leontief–Neumann Dynamic Models and Economic Reality

It is well known, that the basic assumptions of dynamic Leontief models, resp. Neumann-models are very strong; they are gross simplifications of economic reality. The authors do not deny that they have been forced to accept these assumptions in the present stage of research while grinding their teeth. For the sake of some self-assuredness, it is worth while to follow the reasoning as below:

1. Some of our theorems require more restrictive, and some others less restrictive sets of assumptions. Let us consider the most restrictive one at first: time invariant real structure *and* a Neumann-path, serving as the normative path of control. This would be an almost unendurably "bad" model of the economic system if we were going to use it for *planning* the real sphere. By force of the assumptions themselves, we eliminate the substance of the planning problem: the choice of the relative ratios of the sectors and the technologies. The model is much less "bad" for our purposes (for the theoretical analysis of the control mechanism) and precisely of the control mechanism regulating elementary primitive quantitative adaptation.

Not only in our model, but also in reality, the decision makers are inclined to apply in routine-like quantitative adaptation simple recipes, rules of thumb, which do not take into account the structural shifts in the farther economic environment, but react merely to the directly observable signals.

The elementary quantitative adaptation of the Neumann-economy described in the present study operates with the aid of such stock normatives which resemble the form of stock normatives observable in a real economy. The norms are fixed in the form of a quotient of output stock per output or input stock per purchase ("To the production or purchase of how many months does the stock correspond"). We have succeeded in working out such simple forms of norms only for the Neumann-economy up to now.

True, this elementary control mechanism might lead to incorrect decisions if considerable qualitative changes occur, shifts in proportions in the composition of final resources and final consumption, as well as in technology. In such cases the *limits* of the elementary control described in our paper will have been reached. On the one hand other, higher regulators must enter, but on the other hand perhaps the norms must be changed. We feel, however, that this is not a deficiency of the model. We intended to present not a *universal* mechanism equally suited for fulfilling any kind of control task, but a special one capable of fulfilling a few elementary control tasks, and we think it is natural that the control parameters figuring in it, the norms and the speeds of adjustment reaction, must be subjected from time to time to revision.

2. The assumptions about the asymptotic economy are less strong than those treated in paragraph 1. Also, the asymptotic economy is an abstraction, but one standing nearer to reality. The technological changes, shifts in proportions which, as regards their lasting trend, show some definite monotonous regularity, can be described with good approximation as an approach to some basic structure. If, for example, with primary energies there is a monotonous tendency for a growing share of nuclear energy, this can be described as approaching some ratio for nuclear to conventional primary energies.[10]

3. We applied even less restricted sets of assumptions to our two most relevant propositions, to Theorem 3.A on stability, and Theorem 5.A

[10] Regarding a longer historical development, the share of coal first increased when wood was pushed to the background, and then it diminished under the effect of the advance of oil and natural gas, and now, at least transitorily, it will be growing again in the wake of the higher oil and gas prices. Such a nonmonotonous shift in proportions cannot be adeqately described with the aid of the asymptotic assumption.

on viability. Here we assume time-variant asymptotic real structure, but we do not need any restriction on the normative path (apart from feasibility). *The system can be controlled in a decentralized way, and it will be stable and viable even when the normative path is not a Neumann-path.*

That is the reason we could not apply to a larger extent the literature on Neumann-growth and the turnpike theorems. Namely, we posed a different set of *questions* to be answered. Turnpike theorems belong to *growth* theory; our theorems belong to the theory of *control of economic systems*; both being markedly different branches of economics. Turnpike theorists are searching for the solution of a *normative* problem: optimal conditions of growth. Our study, on the contrary, is more a *descriptive–explanatory* investigation. We are exploring elementary control mechanisms which are able to regulate any kind of economies, including those which are *nonoptimal*, i.e., do not move along (or close to) a Neumann-path. To find an answer to this broad question, we cannot restrict our study to the much narrower case of optimal growth paths.

5. Comparison with Other Models

As has been mentioned in the Introduction, this paper has been born from cross-breeding three schools: modeling the control mechanism of the economy, the literature of the Leontief–Neumann models, and the economic application of mathematical control theory. It would lead too far if comparisons were made even with the main models of all three schools. Therefore, the circle will be essentially narrowed down. Exclusively those models will be dealt with which examine the *control mechanism of dynamic economic systems.* (Surprisingly few models can be listed in this category.)

The comparison is made in the form of a table. The columns show the different models, the rows the criteria of the models.

Although the table speaks for itself, some additional remarks are in order.

In columns 1–4 and 7 of the table there are models which are closely related: they are all linked to the Kornai–Martos study. Therefore, they are easily comparable. The studies of Lovell [22] and McFadden [23] set out from rather different questions. Therefore, these have been first adapted to the problem examined by us and only then compared to the others.

Only a single attempt has been made so far at a stochastic analysis of the problem: in the article of Virág. One of the most important tasks is to extend research in this direction.

The article by Kornai and Martos [16, 17] and in its wake Virág [33] and Dancs, Hunyadi, and Sivák [8]) did not insist on a real structure

TABLE I

	Kornai and Martos [16, 17]	Virág [33]	Dancs, Hunyady, and Sivák [8]	Bródy [5]	Lovell [22]	McFadden [23]	Kornai and Simonovits
0 Deterministic 1 Stochastic	0	1	0	0	0	0	0
0 Continuous over time 1 Discrete ove time	0	0	0 and 1	0 and 1	1	1	1
0 Engagement of assets is not represented 1 Engagement of assets is represented	0	0	0	0	1	0	1
0 Structure is constant over time 1 Structure is variable over time	1	1	1	0	0	0	0 and 1
0 Stagnating 1 Growing	1	1	1	0	0	1	1
0 Centralized 1 Decentralized	1	1	0	1	0 and 1	1	1
0 Memory 1 No memory	0	0	0	0 and 1	0	1	1

constant time, but allowed its arbitrary changes in time. The present study applies, on the one hand, a stronger assumption when it sets out from a real structure either constant in time or asymptotic, but on the other hand, it has a compensating merit: the system of norms is very simple in the case of the Neumann-economy; it takes a form similar to the stock norms in real life (stocks as a fixed proportion of production or purchase).

The article by Kornai and Martos handled time as a continuous variable and did not describe any lags. The model of Dancs, Hunyadi, and Sivák and the present paper handle time as a discrete variable, with the advantage that it can be directly simulated by a computer. One of the essential differences between the model of Dancs, Hunyadi, and Sivák and the present one is that in the former the control variables can be determined only in possession of centralized information, while in the latter the system is decentralized also from the point of view of information. The former is not autonomous (vegetative) regulation, the latter is one.

From among the studies listed, the present one is the first which describes pure stock signals. In the Kornai–Martos study and in the related other works, in addition to stocks sales also figured among the signals.

There is no way for a simple confrontation of the "centralized" control schemes with the "decentralized" ones, since this depends, among other things, on how the norms come about. (Are they not determined perhaps by some centralized process?) Therefore, in the comparison the sign 1 is put where, in possession of given norms, both the decision and the information serving as a basis of the decision are strictly decentralized.

In conclusion, we should like to stress that the table also indicates the further direction of research. We have made efforts to try the cases denoted by both 1 and 0, from the viewpoints of every criterion. In addition, we should also like to expand research in directions not shown in table, to include scarce primary resources and alternative technologies, for example.

Two further comments on the comparisons.

There are some important theorems related to *decentralized* control, (e.g., in Aoki [1].) These theorems, however, cannot be applied directly to the discussion of Leontief–Neumann dynamic systems, like the economies described in our paper. Therefore we did not include that part of the literature in our comparative table.

In addition, there is a huge literature on *centralized* control, among others on the so-called tracking problem. Unfortunately, *decentralization* of control raises new difficulties in the mathematical nature of the issues, which exclude direct application of the tracking theorems.

APPENDIX: MATHEMATICAL PROOFS OF THE THEOREMS

Proof of Theorem 1. We shall consider the product balance (1.1) for a normative path:

$$r^*(t) = A(t)\, r^*(t) + w^*(t+1) - w^*(t) + V^*(t+1)1 - V^*(t)1. \tag{6.1}$$

If $w^*(t)$ and $V^*(t)$ are given (and positive), then every $w^*(t+1)$ and $V^*(t+1)$ is *reachable* by a unique positive *decision-pair* $r^*(t)$ and $Y^*(t)$ if

$$w^*(t+1) \geqslant w^*(t) \qquad \text{and} \qquad V^*(t+1) \geqslant V^*(t), \tag{6.2}$$

and

$$B(t)\langle[I - A(t)]^{-1} \cdot [w^*(t+1) - w^*(t) + V^*(t+1)1 - V^*(t)1]\rangle \leqq V^*(t). \tag{6.3}$$

To prove this let us mention that (6.2) assures the positivity of

$$r^*(t) = [I - A(t)]^{-1}[w^*(t+1) - w^*(t) + V^*(t+1)1 - V^*(t)1],$$

the formula provided by (6.1) and R.4. Furthermore, $Y^*(t)$ is uniquely determined by (2.2) and by $r^*(t)$, $V^*(t)$, and $V^*(t+1)$. Its $\bar{J}$-variables are positive, its J-variables are zero.

Finally, $V^*(t) \geqq B(t)\langle r^*(t)\rangle$ is equivalent to (6.3). Since $[I - A(t)]^{-1}$ is positive, $B(t)$ and $V^*(t)$ are $\bar{J}$ positive, sufficiently small increases in all normative stocks are feasible. Q.E.D.

Proof of Theorem 1.N. If the N-path exists, by substituting into the product balance the relations of C.2.N and dividing it by $\lambda_0{}^t$, the following equation is obtained:

$$r_0 = Ar_0 + (\lambda_0 - 1)\langle g\rangle\, r_0 + (\lambda_0 - 1)(B + F)\, r_0\,. \tag{6.1.N}$$

After rearrangement this leads to the eigenvalue–eigenvector problem

$$(1/(\lambda_0 - 1))\, r_0 = (I - A)^{-1}(B + H)\, r_0\,.$$

It is well known (see, e.g., Bródy [4, Appendix 1]) that there exists a unique (r_0, λ_0) which is positive and r_0 satisfies the norm $1'r_0 = 1$. r_0 uniquely determines w_0, v_0, and S_0 according to C.2.N and then also Y_0 is uniquely determined, because of (2.2). Each of v_{ij}^0, s_{ij}^0, and y_{ij}^0 is zero if and only if $(i, j) \in J$.

Proof of Theorem 2.N. From the theory of positive matrices (cf. Morishima [26] and Bródy [4]) the following lemma is known:

LEMMA I. *If U_1, U_2 are irreducible matrices with nonnegative elements of the size $n \times n$, for which $U_1 \geqslant U_2$, then $\sigma(U_1) > \sigma(U_2)$.*

In our case $U_i = (I - A)^{-1}(B + H^{(i)})$ and $\sigma(U_i) = 1/(\lambda_0^{(i)} - 1)$ $(i = 1, 2)$. $H^{(1)} \geqslant H^{(2)}$ implies $U_1 \geqslant U_2$, hence by Lemma I $\sigma(U_1) > \sigma(U_2)$. Since $\lambda_0^{(i)} = 1 + 1/\sigma(U_i)$, $\lambda_0^{(1)} < \lambda_0^{(2)}$. Q.E.D.

The following *monotonicity* property plays a great role in examining both (relative) stability and viability.

LEMMA II. *If $0 < d_j, e_{ij} \leqq 1$ $(1 \leqq i, j \leqq n)$, then*

$$w^{(1)}(0) \geqslant w^{(2)}(0) \qquad \text{and} \qquad V^{(1)}(0) \geqslant V^{(2)}(0)$$

imply the inequalities $w^{(1)}(t) > w^{(2)}(t)$ and $V^{(1)}(t) \geqslant V^{(2)}(t)$ for $t = 1, 2,...$ where the structures and the speeds of adjustment of the systems (1) *and* (2) *are identical, only their initial states differ to the advantage of* (1).

Proof of Lemma II. Let us substitute the normative path into (2.1)–(2.2) and deduct it from the original:

$$\hat{w}_j(t+1) = \hat{w}_j(t) - \sum_{k=1}^{n} \hat{y}_{jk}(t) + \hat{r}_j(t),$$

$$\hat{v}_{ij}(t+1) = \hat{v}_{ij}(t) - a_{ij}(t)\,\hat{r}_j(t) + \hat{y}_{ij}(t).$$

Let us substitute (2.3)–(2.4) into our new equations:

$$\hat{w}_j(t+1) = (1 - d_j)\,\hat{w}_j(t) + \sum_{k=1}^{n} e_{jk}\hat{v}_{jk}(t), \tag{6.4}$$

$$\hat{v}_{ij}(t+1) = d_j a_{ij}(t)\,\hat{w}_j(t) + (1 - e_{ij})\,\hat{v}_{ij}(t). \tag{6.5}$$

From the conditions of Lemma II it follows immediately that the coefficient matrix of the iterative formulas (6.4)–(6.5) of the dimension $(n^2 + n - |J|)$[11] is nonnegative, regular, from which the monotonicity of the "system of deviations" is immediately to be seen. Since the "system of deviations" is a system reduced by means of the normative path, the original system is also monotonous.

As a contradiction, we shall assume that our matrix in (6.4)–(6.5) is reducible, at least for one t. Then, by definition, we can divide our $\bar{J}$-variables into two classes: the first class is referred to as the class of *free* variables, while the second class is referred to as the class of *zero* variables; and at this classification each transformed zero variable remains a zero variable, and the second class is nonempty.

[11] $|J|$ denotes the number of the elements of J.

Let $K = \{j;\ \hat{w}_j(t) \text{ is free}\}$ and $\bar{K} = \{j;\ \hat{w}_j(t+1) = 0 = \hat{w}_j(t)\}$. By (6.4), if $j \in \bar{K}$ and $(j, k) \in \bar{J}$, then $\hat{v}_{jk}(t) = 0$. By (6.5), if $j \in K$ and $(i, j) \in \bar{J}$, then $\hat{v}_{ij}(t)$ is free.

K is nonempty, too, because our matrix is nonzero. Since matrix $A(t)$ is irreducible, there exists at least one $(i, j) \in \bar{J}$, for which $i \in \bar{K}$ and $j \in K$. Thus $\hat{v}_{ij}(t)$ is zero as well as free, which is a contradiction. Q.E.D.

Proof of Theorem 3.A. First we shall prove Theorem 3.A for time-invariant systems.

(i) Let $(\hat{\lambda}, \hat{w}, \hat{V})$ be the normed positive dominant solution of the linear system (6.4)–(6.5). We shall call $1/|\hat{\lambda}|$ the (asymptotic) *convergence rate*. Here the matrix of coefficients is nonnegative and irreducible, that is, according to Frobenius' theorem it has a unique positive dominant solution.

With the aid of the positive dominant solution we can give a simple constructive condition for the (relative) stability and the viability of the system.

To prove stability, we need to prove $\hat{\lambda} < 1$. Since $(\hat{\lambda}, \hat{w}, \hat{V})$, a dominant solution of (6.4)–(6.5), is the unique positive solution, it is sufficient to prove the existence of a positive solution with $\hat{\lambda} < 1$.

Substitute $(\hat{\lambda}, \hat{w}, \hat{V})$ into (6.4)–(6.5). After a straightforward calculation we get the following n-dimensional, nonlinear fix-point problem:

$$\hat{w}_j = \sum_{k=1}^{n} \frac{d_j e_{jk}}{(\hat{\lambda} - 1 + d_j)(\hat{\lambda} - 1 + e_{jk})} a_{jk} \hat{w}_k, \tag{6.6}$$

$$\hat{w}_j > 0, \qquad j = 1, \ldots, n \qquad \text{and} \qquad \hat{\lambda} > 0.$$

Let us consider the following characteristic value–vector problem:

$$\sigma_\lambda w_j = \sum_{k=1}^{n} \frac{d_j e_{jk}}{(\lambda - 1 + d_j)(\lambda - 1 + e_{jk})} a_{jk} w_k, \tag{6.7}$$

$$w_j > 0, \qquad j = 1, 2, \ldots, n \qquad \text{and} \qquad \lambda > 0.$$

We underline that for $\lambda = \hat{\lambda}$ the second problem is reduced to the first one. Let $\mu = \max\{1 - d_j, 1 - e_{jk};\ 1 \leqslant j \leqslant n;\ (j, k) \in \bar{J}\}$.

By $0 < d_j, e_{jk} \leqq 1$, $0 \leqslant \mu < 1$. If $\lambda > \mu$; then each coefficient of the problem is nonnegative and their matrix is irreducible, with spectral radius σ_λ.

At $\lambda = \mu + 0$ at least one coefficient is equal to $+\infty$, hence $\sigma_{\mu+0} = +\infty$, too. At $\lambda = 1$, $\sigma_1 = \sigma(A)$, which is smaller than 1. Since σ_λ is continuous, by the intermediate-value theorem (Bolzano theorem) there exists a $(\hat{\lambda}, \hat{w}, \hat{V})$ with positive elements and $\hat{\lambda} < 1$. Q.E.D.

(ii) To get a necessary and sufficient condition on stability, we shall restrict our analysis to *homogeneous adjustment speeds*. Let $d_j = \delta = e_{jk}$, $1 \leqq j, k \leqq n$, then we get from our fixpoint problem (6.6) the following characteristic value–vector problem:

$$((\lambda - 1 + \delta)^2/\delta^2)w = Aw.$$

Since we have dropped the assumption of *damped adjustment speeds*, we do not know a priori if the unique positive solution is a dominant one. Let $\{\nu_i\}_{i=1}^n$ be the characteristic values of the matrix A and let $(\{\lambda_i^{(h)}\}_{i=1,h=1}^{n,2})$ be the corresponding characteristic values of the system (6.4)–(6.5). Obviously

$$\lambda_i^{(1,2)} = 1 - \delta(1 \pm \nu_i^{1/2}) \qquad 1 \leqq i \leqq n. \tag{6.8}$$

Now $|\hat{\lambda}| = \max\{|\lambda_i^{(h)}| \mid 1 \leqq i \leqq n, h = 1, 2\}$. Evidently, if $\delta \leqslant 0$, then all $|\lambda_i^{(1)}|$'s are greater than or equal to 1, i.e., the system is *unstable*.

$$\text{If } \delta > 1, \text{ then } |\hat{\lambda}(\delta)| = |1 - \delta[1 + \sigma^{1/2}(A)]| \tag{6.9}$$

and this is smaller than unity if and only if $\delta < 2/[1 + \sigma^{1/2}(A)]$. Q.E.D.

Proof of Theorem 3.A (for time-variant systems). It is well known that the stability theorems referring to the linear difference and differential equations can be easily extended from constant matrices to asymptotic ones, as proved by Poincare (cf. Gelfond [12]) and Perron (cf. Bellman [3, Chap. II, Theorem 2]). Relying on them, we state without further proof

LEMMA III. *Let the $z(t+1) = Uz(t)$ first-order linear system of difference equations with constant coefficients be stable, that is $\sigma(U) < 1$. Then there exists an adequately small $\epsilon > 0$ and an adequately large integer T, for which with an arbitrary series of matrices $\{U(t)\}_{t=0}^{\infty}$, the system of equations $\tilde{z}(t) = U(t)\,\tilde{z}(t)$ is equally stable, provided that $\| U(t) - U \| < \epsilon$ for every $t > T$.*

Proof of Theorem 3.N. We return to the Neumann-economy with homogenous adjustment speeds. Since $0 < \delta \leqq 1$ assures stability as well as relative stability, now we turn to $\delta \leqslant 0$ and $\delta > 1$. From (6.8) it follows that in both cases $\hat{\lambda}(\delta) = -1 + [1 + \sigma(A)^{1/2}]\delta$. Relative stability is equivalent to $|\hat{\lambda}| < \lambda_0$, thus $\hat{\lambda}(\delta_{\inf}) = -\lambda_0$ and $\hat{\lambda}(\delta_{\sup}) = \lambda_0$ define the infimum and the supremum of those homogenous adjustment speeds which provide relative stability. The last three equations provide the results. Q.E.D.

Proof of Theorem 4. (i) The proof is rather similar to that of Theorem 2.N, therefore we emphasize only the new aspects.

Considering again (6.7), we see that each coefficient of the matrix is a decreasing function of d_j as well as of e_{jk}, $(j, k) \in \bar{J}$. Hence comparing our two economies we get $\sigma_\lambda^{(1)} > \sigma_\lambda^{(2)}$ for every $\lambda > \mu^{(1)}$. Hence $\hat{\lambda}^{(1)} > \hat{\lambda}^{(2)}$.

(ii) Consequently, the optimal damped adjustment speeds are all maximal, i.e., they are equal to unity.

(iii) Now we drop the assumption of damped adjustment speeds but assume homogenous adjustment speeds. If $\delta < 0$, then the system is unstable. If $0 < \delta \leqq 1$, then we have seen just before that $\delta = 1$ is the optimum. If $\delta > 1$ then (6.9) implies that $|\hat{\lambda}(\delta)| > |\hat{\lambda}(1)|$. Q.E.D.

Remark. It would have been rather cumbersome to apply McFadden's theorem, and anyway the convergence rate would have been much less than our optimum, because the adjustment speeds would have been almost zero.

Proof of Theorem 5.A. It is generally true that, in the case of a relatively stable system of difference equations, if the equilibrium path is at every point of time bounded from below by a positive vector which is constant over time, there exists an initial neighborhood from which we can always set out on a path which is positive all the way along. Q.E.D.

Proof of Theorem 5.N. For the case of the Neumann-economy a constructive method will be given to determine an initial neighborhood that is maximal, in a sense.

We need the following quantities:

$$\omega = \min_{(i,j)\in J} \frac{s_{ij}^0}{b_{ij} d_j \hat{w}_j} \eta \qquad (0 < \eta < 1), \qquad \tilde{\alpha} = \min_{1 \leqq j \leqq n} \frac{w_j{}^0}{\hat{w}_j},$$

$$\alpha = \min\{\omega, \tilde{\alpha}\}, \qquad \beta = \min_{1 \leqq j \leqq n} \frac{r_j{}^0}{d_j \hat{w}_j},$$

$$\gamma = \min_{(i,j)\in J} \frac{s_{ij}^0 - \alpha b_{ij} d_j \hat{w}_j}{\hat{v}_{ij}} \qquad \xi = \min_{(i,j)\in J} \frac{y_{ij}^0}{e_{ij} \hat{v}_{ij}},$$

$$w^{(m)} = w_0 - \alpha \hat{w}, \qquad w^{(M)} = w_0 + \beta \hat{w},$$

$$V^{(m)} = V_0 - \gamma \hat{V} \quad \text{and} \quad V^{(M)} = V_0 + \xi \hat{V}.$$

LEMMA IV. *The initial neighborhood defined with the aid of the inequalities* $w^{(m)} \leqq w(0) \leqq w^{(M)}$ *and* $V^{(m)} \leqq V(0) \leqq V^{(M)}$ *is not empty but viable. The increase of any of the* (α), β, γ, *and* ξ *in* $(w^{(m)})$, $w^{(M)}$, $V^{(m)}$, *and* $V^{(M)}$ *will cause the ceasing of viability right at the start, (if* $\tilde{\alpha} < \omega$*).*

Proof of Lemma IV. Let us pass to the "system of deviations." Then our initial neighborhood is $-\alpha\hat{w} \leqq \hat{w}(0) \leqslant \beta\hat{w}$ and $-\gamma\hat{V} \leqq \hat{V}(0) \leqslant \xi\hat{V}$. Because of the monotonicity, $-\alpha\hat{w}\hat{\lambda}^t \leqq \hat{w}(t) \leqq \beta\hat{w}\hat{\lambda}^t$ and $-\gamma\hat{V}\hat{\lambda}^t \leqq \hat{V}(t) \leqq \xi\hat{V}\hat{\lambda}^t$ for every t. Owing to the relative stability of the system $0 < \hat{\lambda} < \lambda_0$ and, because of the definition,

$$-w_0 \leqq \alpha\hat{w}, \qquad \beta\hat{w} \leqslant \langle d_j^{-1}\rangle\, r_0\,,$$
$$-\alpha B\langle d_j\hat{w}_j\rangle - S_0 \leqq -\gamma\hat{V}, \qquad \xi\hat{V} \leqslant [e_{ij}^{-1}] \otimes Y_0\,.$$

It follows that

$$-w_0\lambda_0{}^t < \hat{w}(t) < \langle d_j^{-1}\rangle\, r_0\lambda_0^t$$

and

$$-B\langle d_j\hat{w}_j(t)\rangle - S_0\lambda_0{}^t < \hat{V}(t) < [e_{ij}^{-1}] \otimes Y_0\lambda_0{}^t \qquad (t > 0),$$

which, according to (2.3) and (2.4) secures, beside the positivity of $w(t)$ and $S(t)$, also the positivity of $r(t)$, $V(t)$, and $Y(t)$ for $t > 0$. Q.E.D.

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